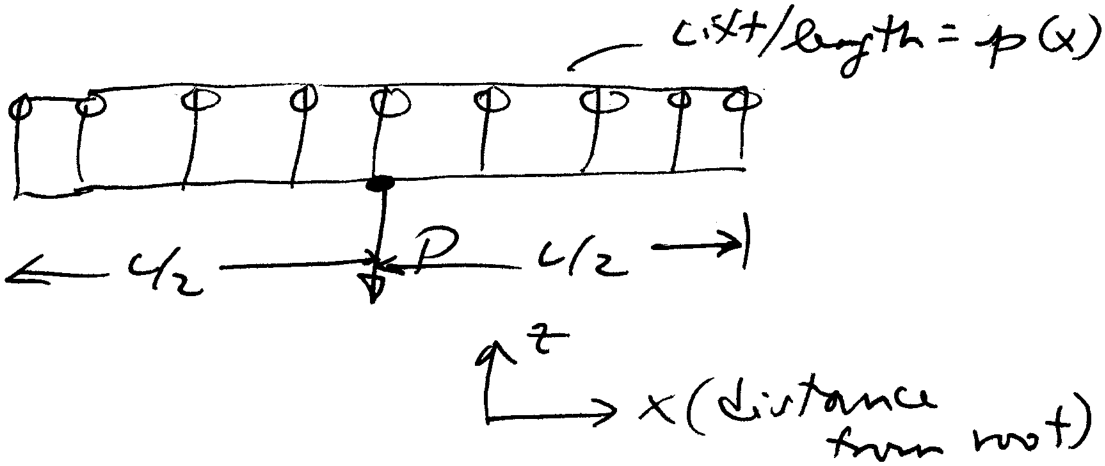
PAL
2/21/08

Unified Engineering Problem Set
Week 4 Spring, 2008

SOLUTIONS

M4.1 We continue considering the wing configuration of M3.2. The case of constant lift load is shown:



We use the results for the internal loads determined in M3.2 for each of the four load cases.

We summarize these from those results for $0 < x < L/2$:

Case 1:

$$q_1(x) = q_0 \qquad q_0 = 2700 \ \frac{lbs}{ft}$$
$$L = 200 \ ft$$
$$F_1(x) = 0$$
$$S_1(x) = q_0\left(x - \frac{L}{2}\right)$$
$$M_1(x) = \frac{q_0}{2}\left(x^2 - Lx + \frac{L^2}{4}\right)$$

Case 2:

$$q_2(x) = q_0\left(1 - \frac{2x}{L}\right) \qquad q_0 = 5400 \ \frac{lbs}{ft}$$
$$L = 200 \ ft$$
$$F_2(x) = 0$$
$$S_2(x) = q_0\left(x - \frac{x^2}{L} - \frac{L}{4}\right)$$
$$M_2(x) = q_0\left(\frac{x^2}{2} - \frac{x^3}{3L} - \frac{Lx}{4} + \frac{L^2}{24}\right)$$

Case 3

$$q_3(x) = q_0\left(1 - \frac{x}{L}\right) \qquad q_0 = 3600 \ \frac{lbs}{ft}$$
$$L = 200 \ ft$$
$$F_3(x) = 0$$
$$S_3(x) = q_0\left(x - \frac{x^2}{2L} - \frac{3L}{8}\right)$$
$$M_3(x) = q_0\left(\frac{x^2}{2} - \frac{x^3}{6L} - \frac{3Lx}{8} + \frac{L^2}{12}\right)$$

Case 4

$$q_4(x) = q_0\left(1 - \frac{4x^2}{L^2}\right) \qquad q_0 = 4048 \ \frac{lbs}{ft}$$
$$F_4(x) = 0$$
$$S_4(x) = q_0\left(x - \frac{4x^3}{3L^2} - \frac{L}{3}\right)$$
$$M_4(x) = q_0\left(\frac{x^2}{2} - \frac{x^4}{3L^2} - \frac{Lx}{3} + \frac{L^2}{16}\right)$$

(a) The axial stress is related to the moment via:

$$\sigma_{xx} = -\frac{M(x)\,z}{I}$$

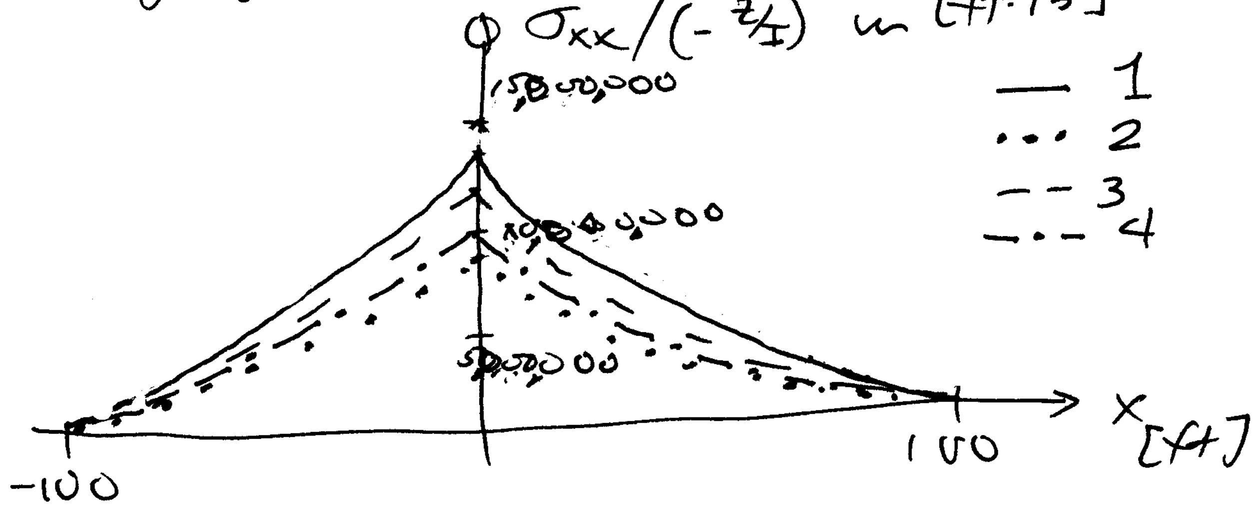
This stress varies at any point x, along the beam with distance from the axis z. We do not know what the cross-section looks like (we just quantify its ability as I), but it does not affect the distribution of σ_{xx} if it is a constant cross-section and thus value. So the maximum stress occurs as far away from the axis that the cross-section goes. The moment is always positive, so the stress will be negative (compressive) for +z and positive (tensile) for −z. This is consistent for a beam that bends up.

The distribution of σ_{xx} with x will be the same as M(x) with the value modified by $-\frac{z}{I}$.

Consider the maximum values of the moment that occurs at the root:

$$M_{1max} = 13.5 \times 10^6 \text{ ft-lb}$$
$$M_{2max} = 9.0 \times 10^6 \text{ ft-lb}$$
$$M_{3max} = 12.0 \times 10^6 \text{ ft-lb}$$
$$M_{4max} = 10.12 \times 10^6 \text{ ft-lb}$$

A plot is the same as for M(x) again being symmetric about x = 0:



$\boxed{\text{Maximum } |\sigma_{xx}| \text{ at } x = 0,\ |z| \text{ maximum}}$

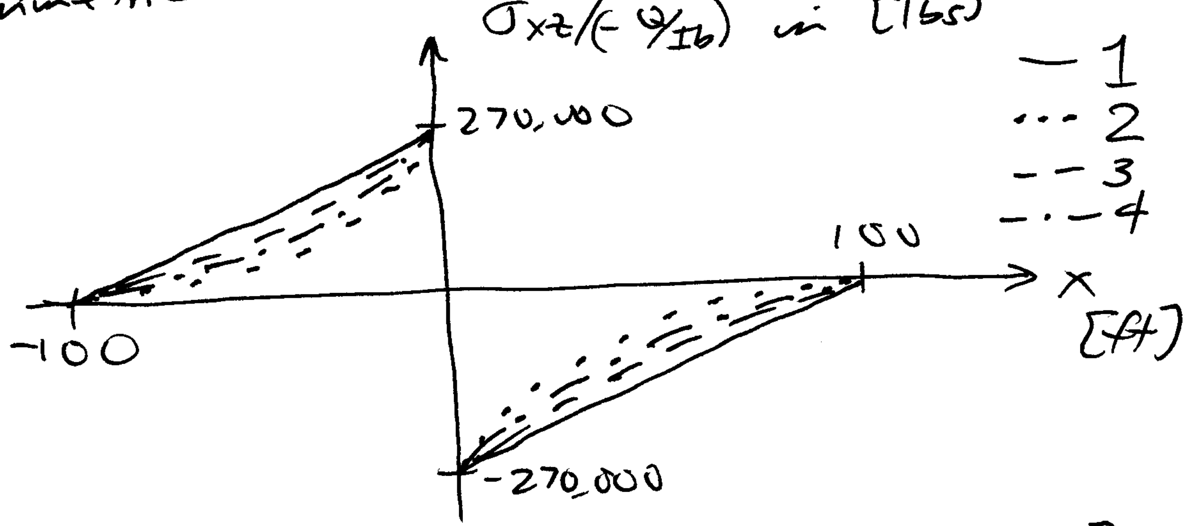
(b) The shear stress is related to shear force via:

$$\sigma_{xz} = -\frac{S(x)\, Q}{Ib}$$

Again, we do not know the specifics of the cross-section, but can say the maximum in z occurs where Q/b is a maximum (the same along x if the cross-section is a constant).

The maximum |value| of the shear is at the root and the same of 270,000 lbs for all cases

The distribution of σ_{xz} with x is the same as $S(x)$ with the value changed by $-Q/Ib$
asymmetric



S in [lbs]

Maximum $|\sigma_{xz}|$ at $x=0$, $\left|\frac{Q}{b}\right|$ maximum

(c) The deflection of a beam, w, is related to the moment via:

$$EI\,\frac{d^2w}{dx^2} = M(x)$$

$M(x)$ is symmetric in x, so $w(x)$ must be as well. We will integrate the various moment equations for $X>0$ only.

Note again that EI does not change in x or for the various cases.

In integrating twice, there will be a need for 2 Boundary Conditions. At the root, the wing is rigidly attached, so:

$$@\ x = 0,\quad w = 0$$

The other B.C. comes from symmetry. Since the wing is continuous, it must have the same slope on each side of the fuselage (at $x = 0$). Due to symmetry the slope must be zero, so:

$$@\ x = 0,\quad \frac{dw}{dx} = 0$$

For each case:

<u>Case 1:</u> $\dfrac{d^2w}{dx^2} = \dfrac{q_0}{2EI}\left(x^2 - Lx + \dfrac{L^2}{4}\right)$ $\quad q_0 = 2700\ \frac{lbs}{ft}$

$$\Rightarrow \frac{dw}{dx}_{①} = \frac{1350\ lbs/ft}{EI}\left(\frac{x^3}{3} - \frac{Lx^2}{2} + \frac{L^2x}{4}\right) + C_1$$

$$@\ x = 0,\ \frac{dw}{dx} = 0 \Rightarrow C_1 = 0$$

$$\Rightarrow w_{①} = \frac{1350}{EI}\ lbs/ft\left(\frac{x^4}{12} - \frac{Lx^3}{6} + \frac{L^2x^2}{8}\right) + C_2$$

$$@\ x = 0,\ w = 0 \Rightarrow C_2 = 0$$

<u>Case 2:</u> $\dfrac{d^2w}{dx^2} = \dfrac{q_0}{EI}\left(\dfrac{x^2}{2} - \dfrac{x^3}{3L} - \dfrac{Lx}{4} + \dfrac{L^2}{24}\right)$ $\quad q_0 = 5400\ \frac{lbs}{ft}$

$$\Rightarrow \frac{dw}{dx}_{②} = \frac{5400}{EI}\ \frac{lbs}{ft}\left(\frac{x^3}{6} - \frac{x^4}{12L} - \frac{Lx^2}{8} + \frac{L^2x}{24}\right) + C_1$$

$$@\ x=0,\ \frac{dw}{dx}=0 \Rightarrow C_1 = 0$$

$$\Rightarrow w_{(2)} = 5400\,\frac{lbs}{ft}\left(\frac{x^4}{24}-\frac{x^5}{60L}-\frac{Lx^3}{24}+\frac{L^2x^2}{48}\right)+C_2$$

$$@\ x=0,\ w=0 \Rightarrow C_2 = 0$$

Case 3:

$$\frac{d^2w}{dx^2} = \frac{q_0}{EI}\left(\frac{x^2}{2}-\frac{x^3}{6L}-\frac{3Lx}{8}+\frac{L^2}{12}\right) \qquad q_0 = 3600\,\frac{lbs}{ft}$$

$$\Rightarrow \frac{dw}{dx}_{(3)} = \frac{3600\,lbs/ft}{EI}\left(\frac{x^3}{6}-\frac{x^4}{24L}-\frac{3Lx^2}{16}+\frac{L^2x}{12}\right)+C_1$$

$$@\ x=0,\ \frac{dw}{dx}=0 \Rightarrow C_1 = 0$$

$$\Rightarrow w_{(3)} = \frac{3600\,lbs/ft}{EI}\left(\frac{x^4}{24}-\frac{x^5}{120L}-\frac{3Lx^3}{48}+\frac{L^2x^2}{24}\right)+C_2$$

$$@\ x=0,\ w=0 \Rightarrow C_2 = 0$$

Case 4:

$$\frac{d^2w}{dx^2} = \frac{q_0}{EI}\left(\frac{x^2}{2}-\frac{x^4}{3L^2}-\frac{Lx}{3}+\frac{L^2}{16}\right) \qquad q_0 = 4048\,\frac{lbs}{ft}$$

$$\Rightarrow \frac{dw}{dx}_{(4)} = \frac{4048\,\frac{lbs}{ft}}{EI}\left(\frac{x^3}{6}-\frac{x^5}{15L^2}-\frac{Lx^2}{6}+\frac{L^2x}{16}\right)+C_1$$

$$@\ x=0,\ \frac{dw}{dx}=0 \Rightarrow C_1 = 0$$

$$\Rightarrow w_{(4)} = \frac{4048\,\frac{lbs}{ft}}{EI}\left(\frac{x^4}{24}-\frac{x^6}{90L^2}-\frac{Lx^3}{18}+\frac{L^2x^2}{32}\right)+C_2$$

$$@\ x=0,\ w=0 \Rightarrow C_2 = 0$$

$\boxed{\text{The maximum value must occur at the tip in all these cases. } (x = \pm 100 \text{ ft})}$

Determine these (all divided by $\frac{1}{EI}$):

Maximum deflections

$$w_{max_{1}} = \frac{1}{EI}(2.70 \times 10^{11}) \; [ft^3 \cdot lb]$$

$$w_{max_{2}} = \frac{1}{EI}(1.80 \times 10^{11}) \; [ft^3 \cdot lb]$$

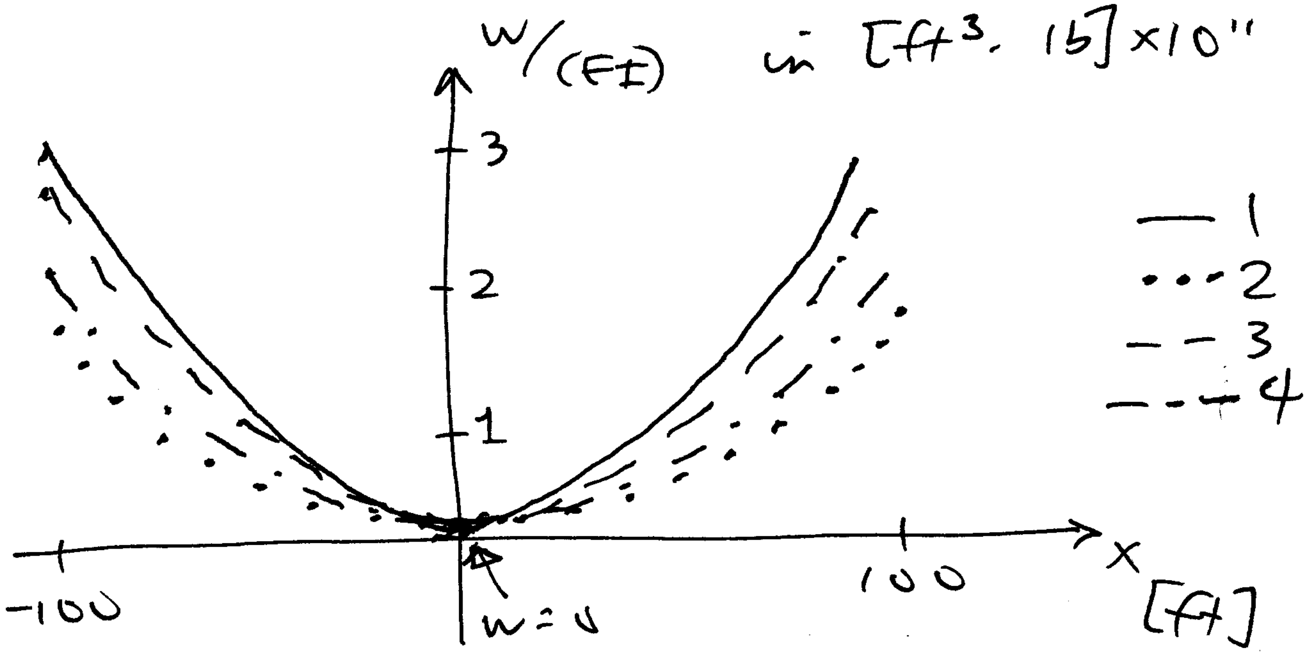
$$w_{max_{3}} = \frac{1}{EI}(2.40 \times 10^{11}) \; [ft^3 \cdot lb]$$

$$w_{max_{4}} = \frac{1}{EI}(2.02 \times 10^{11}) \; [ft^3 \cdot lb]$$

Check units:

$$[L] = \frac{1}{[F/L^2][L^4]}\left[\frac{F}{L}\right][L^4] = [L] \checkmark$$

finally show those:



(Approximate sketches)

M4.2

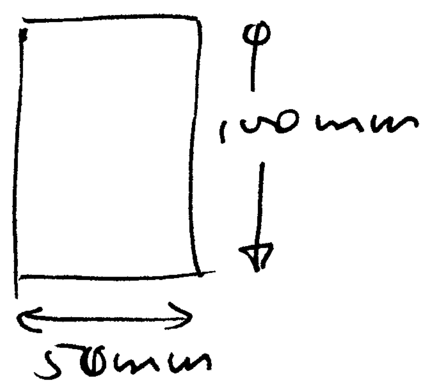
(a) Consider the three cross-sections in sequence in determining the cross-section properties:

A - Area

I - Moment of Inertia

Q - Moment of Area

Solid Rectangle



$$A: (50\text{ mm})(100\text{ mm}) = 5000\text{ mm}^2$$

$$I = \frac{1}{12}bh^3 = \frac{1}{12}(50\text{ mm})(100\text{ mm})^3$$
$$= 4.17\times10^6\text{ mm}^4$$

$$Q = \int_z^{z_{top}} b\,z\,dz$$

This value changes through the thickness. The parameter b is a constant, so this becomes:

$$Q = b \int_{z}^{z_{top}} z \, dz \qquad \text{with } z_{top} = \frac{100\,\text{mm}}{2}$$

$$= 50\,\text{mm} \left[\frac{z^2}{2} \right]_{z}^{50\,\text{mm}}$$

$$\Rightarrow \quad Q = 25\,\text{mm} \left(2500\,\text{mm}^2 - z^2 \right)$$

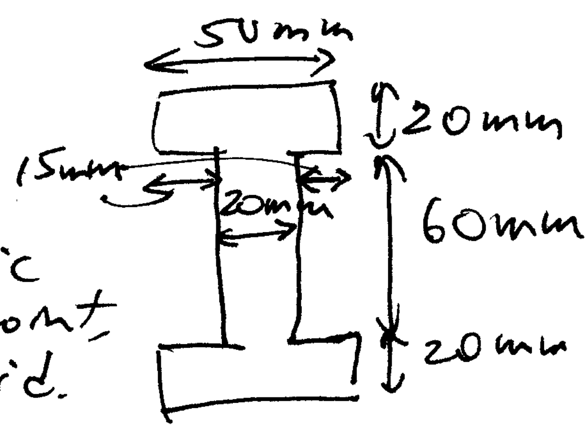
One can see the maximum value is when z^2 is a minimum $\Rightarrow z = 0$ mm

giving $Q_{max} = 62{,}500\,\text{mm}^3$

Summarizing for rectangle:

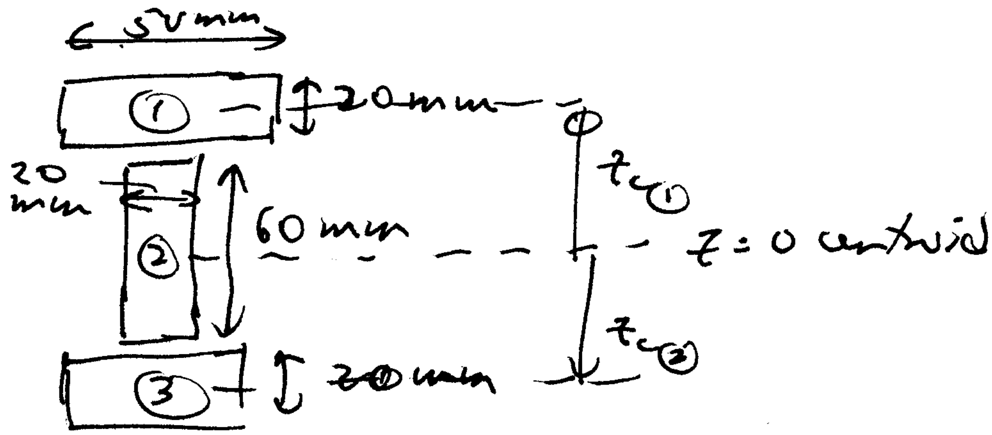
$$\boxed{\begin{aligned} &A = 5000\,\text{mm}^2 \\ &I = 4.17 \times 10^6\,\text{mm}^4 \\ &Q = 25\,\text{mm}\,(2500\,\text{mm}^2 - z^2) \\ &Q_{max} = 62{,}500\,\text{mm}^3 \ @\ z = 0\,\text{mm} \end{aligned}}$$

I-beam



This is symmetric about the centerpoint, so that is the centroid.

Break the cross-sections up into three sections and use the Parallel Axis Theorem. The sections are the 2 flanges and the web:



Assemble the table:

Section	A^* [mm^2]	z_c [mm]	$A z_c^2$ [mm^4]	I_o^* [mm^4]
1	1000	40	1.6×10^6	3.33×10^4
2	1200	0	0	3.6×10^5
3	1000	40	1.6×10^6	3.33×10^4

$* \; I_o = \frac{1}{12} b h^3, \quad A = bh$

Now use the parallel axis theorem:

$$I_{section} = I_o + A z^2$$

and summing:

$$I = \sum_{\#\,sections} (I_o + A z^2)$$

So:

$$I = \{2\times(3.33\times10^{4}) + 3.6\times10^{5} + 2(1.6\times10^{6})\}\ mm^4$$

$$\Rightarrow I = 3.63\times10^{6}\ mm^4$$

$$A = \sum_{\#\ section} A_{section}$$

$$= (2\times1000 + 1200)\ mm^2$$

$$\Rightarrow A = 3200\ mm^2$$

→ In working Q, we have:

$$Q = \int_{z}^{z_{top}} b(z)\, z\, dz \qquad z_{top} = 50\ mm$$

The width changes with z, so each section where width is constant, must be considered separately.

for section ① $\quad 30\,mm < z < 50\,mm$

$$Q_{①} = (50\,mm)\left[\frac{z^2}{2}\right]_{z}^{50\,mm} = 25\,mm\,(2500\,mm^2 - z^2)$$

for section ② $\quad -30\,mm < z < 30\,mm$

Q needs to be divided into the contribution of section ① where b is constant with a value of 50 mm and that of section ② where b has a constant value of 20 mm.

Can express this as:

$$Q = \int_{z}^{z_{top}} b(z)\, z\, dz$$

$$= \int_{30mm}^{50mm} (50mm)\, z\, dz + \int_{z}^{30mm} (20mm)\, z\, dz$$

We get a specific value for section 1 while the contribution from section 2 as we move through the thickness of that section:

$$Q_{2} = 40{,}000\,mm^3 + 20mm\left[\frac{z^2}{2}\right]_{z}^{30mm}$$

$$= 49{,}000\,mm^3 - (10mm)z^2$$

for section 3 proceed the same way. So:

$-50mm < z < -30mm$

$$Q_{3} = \int_{30mm}^{50mm} (50mm)\, z\, dz$$

$$+ \int_{-30mm}^{30mm} (20mm)\, z\, dz$$

$$+ \int_{z}^{-30mm} (50mm)\, z\, dz$$

gives:

$$Q_{3} = 40{,}000\,mm^3 + 0 + 50mm\left[\frac{z^2}{2}\right]_{z}^{-30mm}$$

$$= 62{,}500\,mm^3 - (25mm)z^2$$

Looking at these we find the maximum occurs within section 2 at $z=0$:

$$Q_{max} = 49{,}000\text{ mm}^3$$

Summarizing for I-beam:

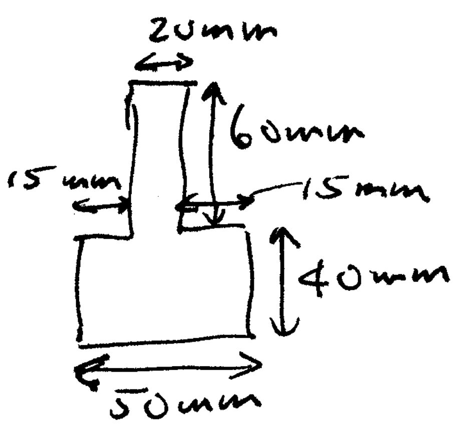
$$A = 3200\text{ mm}^2$$

$$I = 3.63\times10^6\text{ mm}^4$$

$$Q = \begin{cases} 25\text{mm}\,(2500\text{ mm}^2 - z^2) & 30\text{mm} < z < 50\text{mm} \\ 49{,}000\text{ mm}^3 - (10\text{ mm})z^2 & -30\text{mm} < z < 30\text{mm} \\ 62{,}500\text{ mm}^3 - (25\text{mm})z^2 & -50\text{mm} < z < -30\text{mm} \end{cases}$$

$$Q_{max} = 49{,}000\text{ mm}^3 \text{ @ } z = 0\text{ mm}$$

T-beam



This is not symmetric about the centerpoint so we must find the centroid and work from there. Break up the piece into two rectangular sections and work our table after finding the centroid:

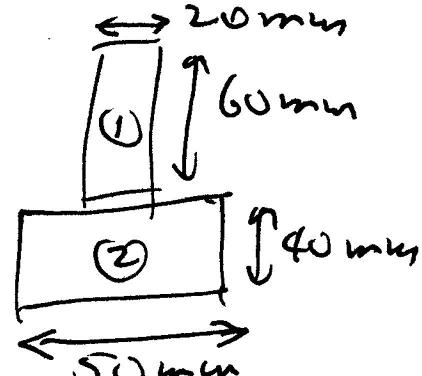
The centroid (center of area) is found via:

$$z_c = \frac{\iint z\,dA}{\iint dA}$$

Define an initial axis system to be midway in the cross-section horizontally (since the cross-section is symmetric in that regard) and at the bottom of the cross-section. (NOTE: This location can be defined anywhere but is chosen here for convenience)

So:

$$z_c = \frac{\iint_{1} z\,dA + \iint_{2} z\,dA}{A_{1} + A_{2}}$$



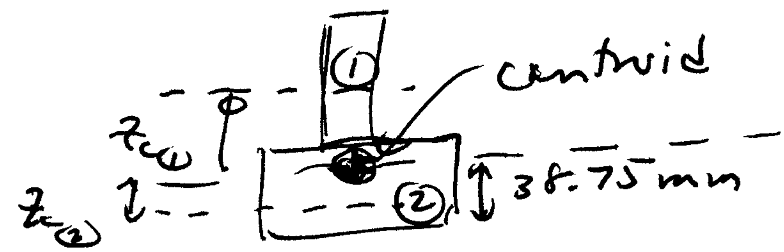
$$\Rightarrow z_c = \frac{(20\text{mm})\int_{40\text{mm}}^{100\text{mm}} z\,dz + (50\text{mm})\int_{0}^{40\text{mm}} z\,dz}{1200\text{ mm}^2 + 2000\text{ mm}^2}$$

$$= \frac{10 z^2\Big]_{40\text{mm}}^{100\text{mm}} + 25\, z^2\Big]_{0}^{40\text{mm}}}{3200\text{ mm}}$$

$$= \frac{10(10{,}000 - 1600)\text{mm} + 25(1600)\text{ mm}}{3200}$$

$\Rightarrow z_c = 38.75\,mm$

Now put the table together:



Section	$A\,[mm^2]$	$z_c\,[mm]$	$A z_c\,[mm^3]$	$A z_c^2\,[mm^4]$	$I_0\,[mm^4]$
①	1200	31.25	37,500	1.172×10^6	3.6×10^5
②	2000	-18.75	-37,500	7.03×10^5	2.67×10^5

as before: $I_0 = \frac{1}{12} b h^3 \quad A = bh$

and:

$z_{c①} = 1.25\,mm + 30\,mm = 31.25\,mm$

1.25 mm: distance to bottom of ① from centroid

30 mm: center of ① within ①

$z_{c②} = -18.75\,mm$

distance (with sign) to center of ② from centroid

to check that the centroid (center of area) was found correctly:

$\sum_{\#\text{ of section}} A z_c = 0$ Yes ✓

Proceed to:

$$A = \sum_{\substack{\#\text{ of} \\ \text{sections}}} A_{\text{section}} = 3200\,\text{mm}^2$$

$$I = \sum_{\substack{\#\text{ of} \\ \text{section}}} \left(I_0 + A z_c^2\right)$$

$$= (3.6\times10^5 + 2.67\times10^5 + 1.172\times10^6 + 7.03\times10^5)\,\text{mm}^4$$

$$\Rightarrow I = 2.50\times10^6\,\text{mm}^4$$

→ In working Q, again need to consider each section separately.

for section ① $\quad 1.25\,\text{mm} < z_c < 61.25\,\text{mm}$

$$Q_{①} = (20\,\text{mm})\left[\frac{z^2}{2}\right]_z^{61.25\,\text{mm}}$$

$$= 10\,\text{mm}\,(3751\,\text{mm}^2 - z^2)$$

for section ② $\quad -38.75\,\text{mm} < z_c < 1.25\,\text{mm}$

$$Q_{②} = \int_{1.25\,\text{mm}}^{61.25\,\text{mm}} (20\,\text{mm})\,z\,dz + \int_z^{1.25\,\text{mm}} (50\,\text{mm})\,z\,dz$$

$$= 37{,}500\,\text{mm}^3 + 39.1\,\text{mm}^3 - (25\,\text{mm})\,z^2$$

A check is that at the bottom ($z = -38.75\,\text{mm}$) Q must go to 0:

$$0 \stackrel{?}{=} 37{,}500\ \text{mm}^3 + 39\ \text{mm}^3 - 37{,}539\ \text{mm}^3$$

YES

Finally, Q_{max} occurs at the centroid (as Q begins to decrease after this point)

$$\Rightarrow Q_{max} = 37{,}539\ \text{mm}^3 \quad \text{at } z_c = 0$$

Summarizing for T-beam:

$$A = 3200\ \text{mm}^2$$

$$I = 2.50 \times 10^6\ \text{mm}^4$$

$$Q = \begin{cases} 37{,}510\ \text{mm}^3 - (10\ \text{mm})z^2 & 1.25\ \text{mm} < z_c < 61.25\ \text{mm} \\ 37{,}539\ \text{mm}^3 - (25\ \text{mm})z^2 & -38.75\ \text{mm} < z_c < +1.25\ \text{mm} \end{cases}$$

$$Q_{max} = 37{,}539\ \text{mm}^3 \ @\ z_c = 0\ \text{mm}$$

(b) Proceed to calculating and comparing I/A for the three cross-sections:

Rectangle: $I/A = 834\ mm^2$
I-beam: $I/A = 1134\ mm^2$
T-beam: $I/A = 781\ mm^2$

The parameter I/A is a measure of the bending efficiency of a cross-section -- how effectively the area is "used" in resisting bending.

The I cross-section is the most efficient (effective). Why? The further away from the centerline the area is, the more effective it is since the contribution is based on z^2.

M 4.3 $\qquad A = 30 \text{ in}^2 \qquad h_{max} = 12 \text{ in}$, $b_{max} = 6 \text{ in}$

(a) We want a cross-section, with these constraints to minimize beam deflection.

Start with:

$$\frac{d^2 w}{dx^2} = \frac{M}{EI}$$

E is constant for all cases.
M is the same for all cases, although it can vary in x.

Thus:
$$w = \frac{1}{E}\frac{1}{I}\int\left(\int M(x)\,dx\right)dx$$

since I does not change with x (along the beam.

So to minimize w, we need to maximize I. Generically:
$$I = \iint z^2\, dA$$

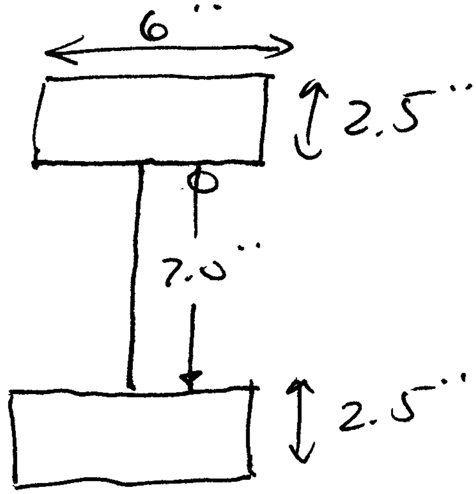
So: $I \propto z^3$

$\rightarrow$ the more you place further from the centroid, the better

We do this via a symmetric cross-section. The maximum occurs for a I-beam. Place the area in the flanges. Maximize the width $\Rightarrow b_{flange} = 6''$

Maximize the distance from the centroid and the thickness of the flanges. Total area $= 30 \text{ in}^2 \Rightarrow$ flange thickness $= 2.5''$ with distance from centroid to center $=$ $6.0'' - 1.125'' = 4.825''$:

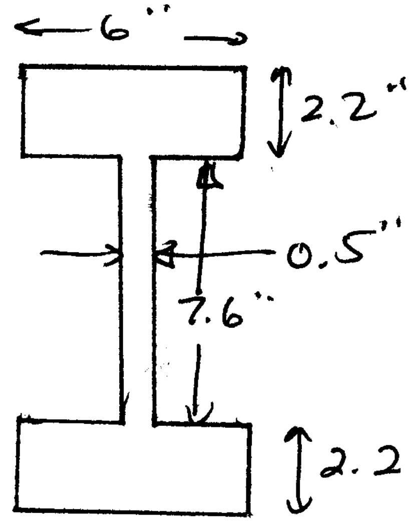
This gives an I-beam that looks like:



but the web has ___zero___ width -- VERY difficult to build. One will need to give finite width to the web and remove that material from the flanges closest to the centroid. Web width will depend on other items.

Choosing a value of 0.5" gives flange area overall reduced by 3.5" reducing each flange thickness by $\frac{1}{2}\frac{3.5''}{6.0''} \approx 0.3''$

Final design is:



(b) We want to minimize the maximum value of σ_{xx}.

Start with:

$$\sigma_{xx} = -\frac{Mz}{I}$$

We cannot change M so:

$$\sigma_{xx} \propto \frac{z}{I}$$

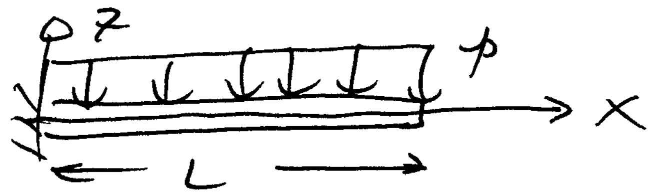
we showed in (a):

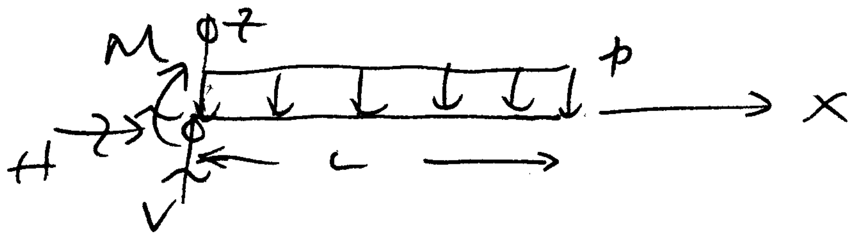
$$\sigma_{xx} \propto \frac{z}{z^3} \propto \frac{1}{z^2}$$

So to get the smallest maximum value, we again want as much area as far away from the centroid as possible.

We follow the same reasoning as in part (a), get an I cross-section and with the same dimensions as in (a).

The maximum value occurs for maximum $z = 6''$. It occurs at the point of the maximum moment. For a cantilevered beam with a constant distributed load:





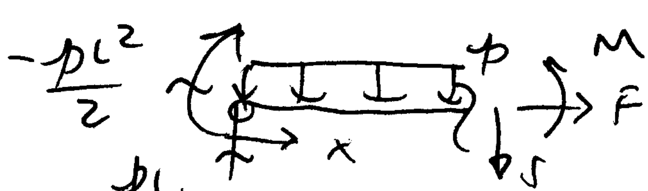
$$\sum F_x = 0 \Rightarrow H = 0$$

$$\sum F_z = 0 \quad \uparrow + \Rightarrow V - pL = 0 \Rightarrow V = pL$$

$$\sum M_0 = 0 \quad \circlearrowleft + \Rightarrow -M - \int_0^L px\,dx = 0$$

$$M = -p\frac{x^2}{2}\Big]_0^L = -\frac{pL^2}{2}$$

Take a cut:



$$\sum M = 0 \circlearrowleft^{+} \Rightarrow M(x) + \frac{pL^2}{2} - pLx + \int_0^x p(x - x')\,dx'$$

gives

$$M(x) = \frac{pL^2}{2} - pLx + \left[pxx' - \frac{1}{2} p \frac{x'}{2} \right]_0^x$$

$$M(x) = p\left(\frac{L^2}{2} - Lx + \frac{1}{2}x^2\right)$$

$\frac{\partial M}{\partial x} = 0$ to find maximum (or minimum)

$$\Rightarrow \quad p(-L + x) = 0$$

gives $x = L$

<u>But</u> this is a minimum of $M = 0$.
Must check the boundary and the maximum magnitude is at the root (as it should be for this cantilevered configuration)

$\Rightarrow$ Maximum $|\sigma_{xx}|$ at $\underset{(x=0)}{\text{root}}$ at $z = \pm 6$ in

(c) We want to minimize the maximum magnitude of σ_{xz}.

Start with:

$$\sigma_{xz} = -\frac{S\,Q(z)}{I\,b(z)}$$

We cannot change $s(x)$ for different cross-sections, so:

$$|\sigma_{xz}| \propto \frac{Q(z)}{I\,b(z)}$$

This gets a bit trickier as Q varies with z and I and Q change with unsymmetric cross-sections as $z_{centroid}$ changes relative to the midpoint of the 12" dimension. The width also changes with z, so looking at the individual contributions, one wants to:

- Minimize $Q(z)$
- Maximize $b(z)$ where $Q(z)$ is maximum.
- Maximize I

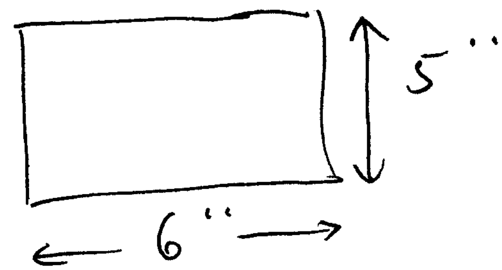
$$Q = \int_z^{z_{top}} b\,z\,dz \Rightarrow \propto z^2$$

so: $$|\sigma_{xz}| \propto \frac{z^2}{z^3 b}$$

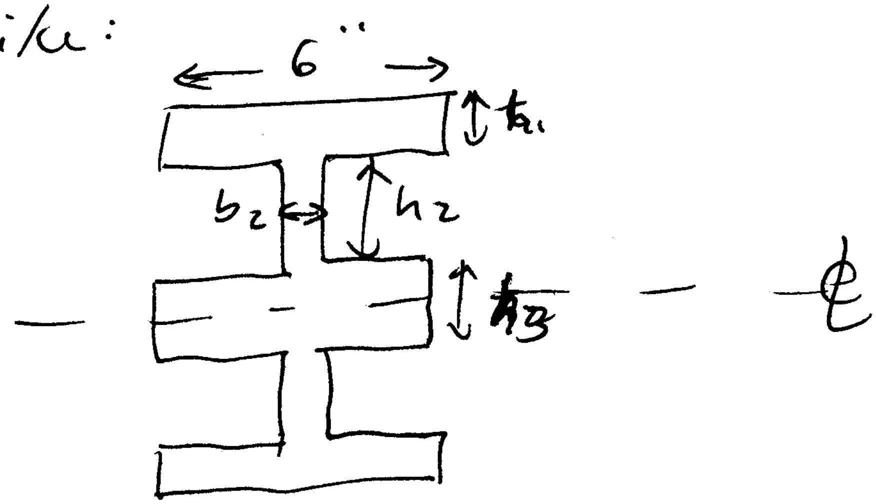
$$\propto \frac{1}{z\,b}$$

One wants a full width of 6" at the point of maximum $Q(z)$ which is at the centroid. One must also keep all other $\frac{Q}{b}(z)$ values lower than this.

This leads to concluding to the need to start from a rectangular cross-section such that:



We now want to maximize I with these constraints. So we want outer flanges as far as possible from the centroid and as wide as possible (6") and connected by webs wide enough to keep σ_{xz} below that at the centroid. So the design will look like:



We must have $6''(2h_1 + h_3) + 2b_2h_2 = 30\,in^2$

and $2(h_1 + h_2) + h_3 = 12''$

use in above: $2h_1 + h_3 = 12'' - 2h_2$

$$72\,\text{in}^2 - 6''(2h_2) + 2b_2h_2 = 30\,\text{in}^2$$

$$\Rightarrow 2h_2(6'' - b_2) = 42\,\text{in}^2 \qquad (A)$$

So: $$\frac{Q_{max}}{6''} = \frac{Q_{@\, z = h_3/2}}{b_2}$$

$$Q_{max} = \tfrac{6''}{2}\left\{36\,\text{in}^2 - (h_3/2 + h_2)^2\right\} + \tfrac{b_2}{2}\left\{(\tfrac{h_3}{2} + h_2)^2 - (h_3/2)^2\right\} + \tfrac{6''}{2}\left\{h_3/2\right\}^2$$

$Q_{@\, z = h_3/2}$ = first two parts of this.

So:

$$\frac{1}{2}\left\{36\,\text{in}^2 - (h_3/2 + h_2)^2\right\} + \frac{b_2}{12''}\left\{\left(\frac{h_3}{2} + h_2\right)^2 - \left(\frac{h_3}{2}\right)^2\right\} + \frac{1}{8}h_3^2 = \frac{6''}{2b_2}\left\{36\,\text{in}^2 - (h_3/2 - h_2)^2\right\} + \frac{1}{2}\left\{h_2^2\right\}$$

use (A): $$h_2 = \frac{21\,\text{in}^2}{6'' - b_2}$$

Thus:

$$\frac{1}{2}\left\{36\,\text{in}^2 - \left(\frac{h_3}{2} + \frac{21\,\text{in}^2}{6'' - b_2}\right)\right\} + \frac{b_2}{12''}\left(\frac{21\,\text{in}^2}{6'' - b_2}\right)^2 + \frac{1}{8}h_3^2 = \frac{6''}{2b_2}\left\{36\,\text{in}^2 - \left(\frac{h_3}{2} - \frac{21\,\text{in}^2}{6'' - b_2}\right)\right\}^2 + \frac{1}{2}\left(\frac{21\,\text{in}^2}{6'' - b_2}\right)^2$$

We work this equation to maximize I subject to this constraint.

$$I = \sum (I_0 + A z^2)$$

$$= 2 (6'') \frac{h_1^3}{12} + 2(6'') h_1 (6'' - h_1/2)^2$$

$$+ 2(b_2) \frac{h_2^3}{12} + 2 (b_2 h_2)$$

$$+ \frac{6''}{12} h_3^3$$

using (A):

$$I = (1'') h_1^3 + (12'') h_1 (6'' - h_1/2)^2 + \frac{1}{6} b_2 \left(\frac{21 \text{ in}^2}{6'' - b_2}\right)^3$$

$$+ \frac{(42 \text{ in}^2) b_2}{6'' - b_2} + \left(\frac{1}{2}\right)'' h_3^3$$

Work this with the constraints to get the value of the dimensions of the cross-section.

The maximum value occurs at $z = 0''$ and from the earlier diagrams (of (b)), maximum shear, $S(x)$, is at the root. Thus, at $x = 0$.